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Making holes in the product of two smooth dendroids

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Abstract

A *continuum* is a non-degenerate connected compact metric space. Let X and Y be continua such that $X \times Y$ is unicoherent. An element $(p, q) \in X \times Y$ makes a hole in $X \times Y$ if $(X \times Y) - \{(p, q)\}$ is not unicoherent. In this paper, we characterize the elements $(p, q) \in X \times Y$ such that (p, q) makes a hole in $X \times Y$, where X and Y are smooth dendroids.

Keywords: Continuum, smooth dendroid, unicoherence, make a hole, property (b)

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1. Introduction

Unicoherence is an important topological property. It arose during the study of topological properties of the Euclidan spaces, cubes, spheres, real projective spaces, Hilbert cube and non-separating Peano subcontinuum of the 2-sphere. Since its introduction, this concept has seen a increasing interest among topologist having as result a lot of papers in the literature related to it. To the present day, there are unsolved question about unicoherence. Intuitively, we can say that a connected space will be unicoherent if it has no "holes". The unicoherence is not a hereditary property. Based in this last fact, our interest

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is aimed at characterizing the points of a unicoherent space such that its complement, as a subspace of the original space, is also unicoherent. In intuitive terms, the points of this class make a "hole" in the space. The classification of the points that make a hole in a unicoherent space has been used to distinguish spaces, especially in hyperspaces of continua (see [1] and [2]). Naturally, one can wonder about the classification of the points that make a hole in other topological structures.

In formal terms, a connected topological space Z is *unicoherent* if whenever $Z = A \cup B$, where A and B are connected closed subsets of Z, we have $A \cap B$ is connected, and an element z of a unicoherent space Z *makes a hole in* Z if $Z - \{z\}$ is not unicoherent.

In this paper, we are interested in the following problem.

Problem. Let X and Y be continua such that $X \times Y$ is unicoherent. For which elements $(p, q) \in X \times Y$, (p, q) makes a hole in $X \times Y$.

Theorems in Section 4 in the current paper give a partial solution to our problem, namely, when X and Y are smooth dendroids.

The use of continuous function of a given space to the unit circumference in the Euclidean plane has been the most powerful tool to study unicoherence. The known results until now of this technique are not easily applicable for the case of the space that results from removing a point to the product of two smooth dendroids. This leads us to introduce a completely novelty method using this class of continuous functions to show that a metric space is unicoherent.

2. Notation and auxiliary results

The symbols $\mathbb{R}$ and $\mathbb{N}$ represent the set of real numbers and the set of positive integers, respectively.

A point z of a connected topological space Z is called *cut point* (*non-cut point*) if $Z - \{z\}$ is not connected (connected). The set $Cut(Z)$ consists of all cut points of Z and let $NCut(Z) = Z - Cut(Z)$.

The subspace $[0,1]$ of the real line $\mathbb{R}$ with the usual topology is denoted by I. An *arc* is any space homeomorphic to I.

By an *end point* of an arcwise connected topological space Y, we mean end point in the classical sense, which means a point that is a non-cut point of any arc in Y that contains it. The set of all end points of Y is denoted by $E(Y)$.

The word *map* stands for a continuous function.

Given a topological space Y, a subspace X of Y is said to be a *deformation retract of* Y if there exists a map $h : Y \times I \to Y$ such that $h(y, 1) = y$ for every $y \in Y$, $h(Y \times \{0\}) = X$, and $h(x, 0) = x$ for every $x \in X$.

A topological space Y is said to be *contractible* if there exists $y \in Y$ satisfying that $\{y\}$ is a deformation retract of Y. In this case, the map h is called *contraction from* Y *to* $\{y\}$.

Convention: when the domain of a sequence in a metric space X is understood from the context, or is not relevant to the discussion, for sake of simplicity, we write $\langle w_k \rangle$ instead of $\{w_k\}_{k=1}^{\infty}$. For a metric space X, let $\mathbb{S}(X)$ be the set of all pairs $(\langle w_k \rangle, w_0)$ where $\langle w_k \rangle$ is a sequence in X converging to $w_0 \in X$.

The result bellow is well know.

Proposition 2.1. *Let X and Y be metric spaces, let $x_0 \in X$ and let $f : X \to Y$ be a function. Then, f is continuous at x_0 if and only if for each $(\langle x_n \rangle, x_0) \in \mathbb{S}(X)$ there exists a subsequence $\langle f(x_{n_k}) \rangle$ of $\langle f(x_n) \rangle$ such that $(\langle f(x_{n_k}) \rangle, f(x_0)) \in \mathbb{S}(Y)$.*

A map f from a connected topological space Z into the unit circumference centred at the origin in the Euclidean plane S^1 *has a lifting* if there exists a map $h : Z \to \mathbb{R}$ such that $f = \exp \circ h$, where exp is the exponential map of $\mathbb{R}$ onto S^1 defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. A connected topological space Z *has property (b)* if each map from Z into S^1 has a lifting.

The next results appear in the literature, we present them due that they will be used frequently in our main theorems.

Proposition 2.2. *[3, Proposition 9, p. 2001] Let Z be a topological space. If Z is contractible, then Z has property (b).*

Theorem 2.3. *[4, Théorème 6', p. 168] Let Z be a connected metric space. If Z has property (b), then Z is unicoherent.*

Theorem 2.4. *[5, Theorem 4, p. 407] Let Z be a connected topological space, let $z_0 \in Z$, let $f : Z \to S^1$ be a map and let $t \in \exp^{-1}(f(z_0))$. If f has a lifting, then there exists a map $h : Z \to \mathbb{R}$ such that $f = \exp \circ h$ and $h(z_0) = t$.*

The next result is obtained immediately from [6, (3), p. 64]

Proposition 2.5. *Let X be a connected metric space and let $f : X \to S^1$ be a map. If $h_1, h_2 : X \to \mathbb{R}$ are liftings of f and there exists $x_0 \in X$ such that $h_1(x_0) = h_2(x_0)$, then $h_1 = h_2$.*

The property (b) is a topological property and each arc has property (b). Both facts will be used repeatedly without mentioning why is true throughout this paper.

Theorem 2.6. *[4, Théorème 3', p. 168] Let Z be a connected metric space. If there exist closed subsets A and B of Z having property (b) such that $A \cap B$ is connected and $Z = A \cup B$, then Z has property (b).*

The symbol F_H denotes the *harmonic fan*, that is $F_H = \bigcup\{J_k : k \in \mathbb{N} \cup \{0\}\}$, where $J_0 = \{(t, 0) : t \in I\}$ and $J_k = \{(t, \frac{t}{k}) : t \in I\}$ are contained in $\mathbb{R}^2$ for each $k \in \mathbb{N}$. Given $(l, r) \in (\mathbb{N} \cup \{0\}) \times I$, define $J_l(r) = \{(t, u) \in J_l : t \leq r\}$.

Given a continuum X, we define its hyperspace $C(X)$ as the space of all subcontinua of X endowed with the Hausdorff metric (see [7, p. 9]).

Concerning to the convergence of a sequence in $C(X)$, we will use the following equivalence without mentioning explicitly: if $\langle A_k \rangle$ is a sequence in $C(X)$, then $x \in \lim A_k$ if and only if there exists a sequence $\langle x_k \rangle$ satisfying that $\lim x_k = x$ and $x_k \in A_k$ for each $k \in \mathbb{N}$.

A *Whitney map for* $C(X)$ (see [7, p. 105]) means a map $\mu : C(X) \to I$ that satisfies the following conditions:

- For any $A, B \in C(X)$ such that $A \subseteq B$ and $A \neq B$, $\mu_X(A) < \mu_X(B)$
- $\mu(\{x\}) = 0$ for every $x \in X$,
- $\mu(X) = 1$.

For any continuum X, by [7, Theorem 13.4, p. 107], there exists a Whitney map for $C(X)$.

A *dendroid* is an arcwise connected, hereditarily unicoherent continuum. Let X be a dendroid. The symbol xy denote the unique arc from x to y, for each pair of elements x, $y \in X$ such that $x \neq y$ and $xy = \{x\}$ when $x = y$.

A dendroid Z is *smooth at* v if for each $(\langle a_n \rangle, a) \in \mathbb{S}(Z)$, then $(\langle va_n \rangle, va) \in \mathbb{S}(C(Z))$. A continuum Z is a *smooth dendroid* if it is a dendroid and there exists a point v in Z such that Z is smooth at v. For sake of simplicity, we say that a pair (Z, v) is a smooth dendroid provided that Z is a smooth dendroid at v.

3. Results auxiliaries

We define an auxiliary function which will be useful in proofs of the next results.

Let (X, v) a smooth dendroid and fix μ a Whitney map for $C(X)$. Define$g_X : X \times I \to X$ by $g_X(x, t)$ is the only point of vx such that $\mu(vg_X(x, t)) = t\mu(vx)$.

Lemma 3.1. *Let (X, v) a smooth dendroid. Then g_X satisfies each one of the following conditions.*

(3.1.1) *g_X is well defined.*

(3.1.2) *g_X is continuous.*

(3.1.3) *If $x \in X - \{v\}$ and $g_X(x, t) = g_X(x, s)$, then $t = s$.*

(3.1.4) *For each $x \in X$, $g_X(x, 0) = v$. Moreover, if $(x, t) \in (X - \{v\}) \times I$, then $g_X(x, t) = v$ if only if $t = 0$.*

(3.1.5) *For each* $x \in X$, $g_X(x,1) = x$. *Moreover, if* $(x,t) \in (X - \{v\}) \times I$, *then* $g_X(x,t) = x$ *if only if* $t = 1$.

(3.1.6) *For each* $(x,t) \in X \times I$, $g_X(\{x\} \times [0,t]) = vg_X(x,t)$.

Proof. First, for each $t \in I$, from the inclusion $g(v,t) \in vv = \{v\}$, it follows that $g(v,t) = v$. Now, let $(x,t) \in (X - \{v\}) \times I$ be arbitrary. Note that $\mathcal{A} = \{vz : z \in vx\}$ is an arc in $C(X)$ whose end points are $\{v\}$ and $\{vx\}$. Since $0 = \mu(\{v\}) \leq t\mu(vx) \leq \mu(vx)$, by the continuity of the one-to-one map $\mu|_{\mathcal{A}}$, there exists an unique point $g_X(x,t) \in vx$ such that $\mu(vg_X(x,t)) = t\mu(vx)$. The proof of (3.1.1) is complete.

Applying Proposition 2.1, we are going to show that g_X is continuous at each point of $X \times I$. Let $(x_0,t_0) \in X \times I$ be arbitrary. Let $(\langle(x_k,t_k)\rangle,(x_0,t_0)) \in \mathbb{S}(X \times I)$. We may assume that there exists $y_0 \in X$ such that $(\langle g_X(x_k,t_k)\rangle, y_0) \in \mathbb{S}(X)$. Now, since $g_X(x_k,t_k) \in vx_k$ for each $k \in \mathbb{N}$ and $(\langle vx_k\rangle, vx_0) \in \mathbb{S}(C(X))$, we obtain that $y_0 \in vx_0$. By the continuity of μ and fact that X is smooth at v, we have $\mu(vy_0) = \lim \mu(vg_X(x_k,t_k)) = \lim t_k\mu(vx_k) = t_0\mu(vx_0)$. Then, $g_X(x_0,t_0) = y_0$. This finishes the proof of (3.1.2).

Next, we shall argue (3.1.3). Our assumptions guarantee that $\mu(vg_X(x,t)) = \mu(vg_X(x,s))$ and $\mu(vx) > 0$. Hence, by the definition of g_X, we deduce that $t\mu(vx) = s\mu(vx)$. This implies that $t = s$.

Observe that the first part of (3.1.4) and of (3.1.5) is a consequence of the definition of g_X and the second part of both follows from (3.1.3).

In order to show (3.1.6), let $(x,t) \in (X - \{v\}) \times I$ be arbitrary. Hence, $\mu(vx) > 0$. First, let $s \in [0,t]$. Then $g_X(x,s), g_X(x,t) \in vx$ satisfy that $\mu(vg_X(x,s)) = s\mu(vx)$ and $\mu(vg_X(x,t)) = t\mu(vx)$. So, since either $vg_X(x,s) \subseteq vg_X(x,t)$ and $vg_X(x,t) \subseteq vg_X(x,s)$, by the choice of s, we conclude that $vg_X(x,s) \subseteq vg_X(x,t)$. This implies that $g_X(x,s) \in vg_X(x,t)$. Hence, from the continuity of g_X (see (3.1.2)), it follows that $g_X(\{x\} \times [0,t])$ is a subcontinuum of the arc $vg_X(x,t)$ containing its end points $g_X(x,0) = v$ and $g_X(x,t)$. Then $g_X(\{x\} \times [0,t]) = vg_X(x,t)$. Clearly, (3.1.6) holds whenever $x = v$.

The map g_X will be used constantly in this paper without mentioning its definition explicitly.

As a consequence of Lemma 3.1 and Proposition 2.2, we have the following result.

Corollary 3.2. *Let X be a smooth dendroid. Then X is contractible and so X has property (b).*

The continuum F_H is a smooth dendroid and hence F_H has property (b). This fact will be used repeatedly throughout this paper.

Theorem 3.3. *Let X and Y be connected metric space having property (b) and let $(x, y) \in X \times Y$. Then $(X \times \{y\}) \cup (\{x\} \times Y)$ has property (b).*

Proof. Since property (b) is a topological property, we obtain that $X \times \{y\}$ and $\{x\} \times Y$ have property (b). Now, by Theorem 2.6, we deduce that $(X \times \{y\}) \cup (\{x\} \times Y)$ has property (b).

In order to give necessary and sufficient conditions to any metric space have property (b), we introduce the following notions.

For a family $\mathcal{V}$ of subsets of X, a map φ from any topological space into X is called *monotone with respect to* $\mathcal{V}$ provided that for each $V \in \mathcal{V}$, $\varphi^{-1}(V)$ is connected.

Let $\mathcal{U}$ be a covering of a connected metric space X. Then, X is said to be *$\mathcal{U}$-covered with respect property* (b) provided that each element of $\mathcal{U}$ has property (b), there exists a connected closed subset M of X having property (b) such that $M \cap U$ is connected and non-empty for all $U \in \mathcal{U}$ and if $U, V \in \mathcal{U}$ such that $U \cap V \neq \emptyset$, then there exists a connected subset $L(U, V)$ of X having property (b) and $L(U, V)$ fulfils each one of the following conditions $U \cap V \subseteq L(U, V)$, $(U \cap M) \cup (V \cap M) \subseteq L(U, V) \cap M$, the sets $L(U, V) \cap U$, $L(U, V) \cap V$ and $L(U, V) \cap M$ are non-empty connected subsets of X. For $(\langle x_k \rangle, x_0) \in \mathbb{S}(X)$, the space X is said to be *$\mathcal{U}$-Maya space at* $(\langle x_k \rangle, x_0)$, if there exist a subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcap \mathcal{V} \neq \emptyset$ and $\{x_k : k \in \mathbb{N} \cup \{0\}\} \subseteq \bigcup \mathcal{V}$, a Hausdorff space F

having property (b) and a map $\varphi : F \to X$ which is monotone whit respect to $\mathcal{V}$ fulfilling $\varphi^{-1}(\bigcap \mathcal{V}) \neq \emptyset$ and some $(\langle y_k \rangle, y_0) \in \mathbb{S}(F)$ satisfies that $\varphi(y_k) = x_k$ for each $k \in \mathbb{N} \cup \{0\}$. The space X is said to be $\mathcal{U}$-Maya space if and only if X is $\mathcal{U}$-Maya space at each $(\langle x_k \rangle, x_0) \in \mathbb{S}(X)$.

Lemma 3.4. *Let X be metric connected space and let $\mathcal{U}$ be a covering of X. If $U \in \mathcal{U}$ has property (b), then X is $\mathcal{U}$-Maya space at each $(\langle x_k \rangle, x_0) \in \mathbb{S}(U)$.*

Proof. Let $(\langle x_k \rangle, x_0) \in \mathbb{S}(U)$ be arbitrary. Consider $\mathcal{V} = \{U\}$, $F = U$ and $\varphi : F \to X$ be the inclusion map. Notice that F has property (b), $\bigcap \mathcal{V} \neq \emptyset$, $\{x_k : k \in \mathbb{N} \cup \{0\}\} \subseteq \bigcup \mathcal{V}$, $(\langle x_k \rangle, x_0) \in \mathbb{S}(F)$ satisfies that $\varphi(x_k) = x_k$ for each $k \in \mathbb{N} \cup \{0\}$ and φ is monotone with respect to $\mathcal{V}$ such that $\varphi^{-1}(\bigcap \mathcal{V}) \neq \emptyset$. So, $\mathcal{V}$, F, φ and $\langle x_k \rangle$ satisfy the required properties.

Lemma 3.5. *A connected metric space X has property (b) if and only if there exists a covering $\mathcal{U}$ of X such that X is $\mathcal{U}$-covered with respect property (b) and X is a $\mathcal{U}$-Maya space.*

Proof. The necessity follows from the fact that X is $\{X\}-$covered with respect property (b) and X is a $\{X\}-$Maya space.

Suppose that exists a covering $\mathcal{U}$ of X such that X is $\mathcal{U}$-covered with respect property (b) and X is a $\mathcal{U}$-Maya space. We will show that X has the property (b). To this end, let $f : X \to S^1$ be a map.

Since X is $\mathcal{U}$-covered with respect property (b), there exists a connected closed subset M of X fulfilling the conditions in the definition. Then M has property (b), therefore there exists a map $\gamma : M \to \mathbb{R}$ such that $f|_M = \exp \circ \gamma$.

Now, for each $U \in \mathcal{U}$, let $z_U \in U \cap M$. The assumption each $U \in \mathcal{U}$ has property (b), Theorem 2.4 and the equality $f|_M = \exp \circ \gamma$ guarantee the existence of a map $\beta_U : U \to \mathbb{R}$ in such way $f|_U = \exp \circ \beta_U$ and $\beta_U(z_U) = \gamma(z_U)$.

Define $\beta : X \to \mathbb{R}$ by $\beta(x) = \beta_U(x)$ if $x \in U$. To see that β is well defined, let $x \in X$ be arbitrary and let $U, V \in \mathcal{U}$ be such that $x \in U \cap V$. As a consequence of the fact that $U \cap V \neq \emptyset$ there exists a connected subset $L(U, V)$ of X having property (b) and satisfying the required properties of the definition. Denote $L(U,V)$ by L. Fix $a \in L \cap M$. Applying Theorem 2.4, since $f(a) = \exp \circ \gamma(a)$ there exists a map $\lambda : L \to \mathbb{R}$ fulfilling $f|_L = \exp \circ \lambda$ and $\lambda(a) = \gamma(a)$. Now, let us argue that $\lambda(x) = \beta_U(x) = \beta_V(x)$.

Since $L \cap M$ is connected, $\gamma(a) = \lambda(a)$ and $\exp \circ (\gamma|_{L\cap M}) = f|_{L\cap M} = \exp \circ (\lambda|_{L\cap M})$ the equality $\gamma|_{L\cap M} = \lambda|_{L\cap M}$ follows from Proposition 2.5. This and the inclusions $z_U \in U \cap M \subseteq L \cap M$ imply $\lambda(z_U) = \gamma(z_U)$. Now, by the choice of β_U, it follows that $\beta_U(z_U) = \gamma(z_U) = \lambda(z_U)$. Observe that $\exp \circ (\lambda|_{L\cap U}) = f|_{L\cap U} = \exp \circ (\beta_U|_{L\cap U})$. Now, invoke Proposition 2.5 to prove that $\lambda|_{L\cap U} = \beta_U|_{L\cap U}$. Our assumptions ensure that $x \in U \cap V \subseteq L$ and so $\lambda(x) = \beta_U(x)$. Similarly, we deduce $\lambda(x) = \beta_V(x)$. In conclusion $\beta_U(x) = \beta_V(x)$.

From the definition of β, it follows that $f = \exp \circ \beta$.

To check the continuity of β, using Proposition 2.1, we are going to show that β is continuous at each point of X. Let $x_0 \in X$ be arbitrary and let $(\langle x_k \rangle, x_0) \in \mathbb{S}(X)$. It suffices to prove that there exists a subsequence $\langle x_{k_j} \rangle$ of $\langle x_k \rangle$ such that $(\langle \beta(x_{k_j}) \rangle, \beta(x_0)) \in \mathbb{S}(\mathbb{R})$.

By hypothesis we deduce that X is a $\mathcal{U}$-Maya space at $(\langle x_k \rangle, x_0)$, so there exist a subset $\mathcal{V}$ of $\mathcal{U}$, a Hausdorff space F having property (b), $(\langle y_k \rangle, y_0) \in \mathbb{S}(F)$ and a map $\varphi : F \to X$ which is monotone with respect to $\mathcal{V}$ fulfilling the conditions in the definition.

The continuity of φ and of f implies that $f \circ \varphi : F \to S^1$ is continuous. Fix $c \in \varphi^{-1}(\bigcap \mathcal{V})$. Since F has the property (b), by Theorem 2.4, there exists a map $h : F \to \mathbb{R}$ such that $f \circ \varphi = \exp \circ h$ and $h(c) = \beta \circ \varphi(c)$.

Now, let us argue $h(y_k) = \beta \circ \varphi(y_k) = \beta(x_k)$ for all $k \in \mathbb{N} \cup \{0\}$.

Let $k \in \mathbb{N} \cup \{0\}$ be arbitrary. Choose $V \in \mathcal{V}$ in such way $x_k \in V$. Then $\varphi^{-1}(V)$ is connected. Now, since $\exp \circ h = \exp \circ (\beta \circ \varphi)$, we obtain that $\exp \circ (h|_{\varphi^{-1}(V)}) = \exp \circ (\beta \circ \varphi|_{\varphi^{-1}(V)}) = \exp \circ \beta|_V \circ \varphi = \exp \circ \beta_V \circ \varphi$. Finally, the inclusion $c \in \varphi^{-1}(V)$ and Proposition 2.5, imply that $h|_{\varphi^{-1}(V)} = \beta_V \circ \varphi$. Thus, $h(y_k) = \beta_V \circ \varphi(y_k) = \beta_V(x_k) = \beta(x_k)$. To conclude, observe that the continuity of h guarantees that $\lim \beta(x_k) = \beta(x_0)$.

Theorem 3.6. *Let X and Y be connected metric spaces and let $(p,q) \in X \times Y$. If there exists a covering $\mathcal{U}$ of $(X \times Y) - \{(p,q)\}$ such that $(X \times Y) - \{(p,q)\}$ is $\mathcal{U}$-covered with respect property (b) and $(X \times Y) - \{(p,q)\}$ is a $\mathcal{U}$-Maya space, then $(X \times Y) - \{(p,q)\}$ has property (b).*

Proof. The connectedness of $(X \times Y) - \{(p,q)\}$ follows from [8, Lemma 2.2, p. 26]. Now, our assumptions and Lemma 3.5 ensure that $(X \times Y) - \{(p,q)\}$ has property (b).

Corollary 3.7. *Let X and Y be continua such that $X \times Y$ is unicoherent and let $(p,q) \in X \times Y$. If there exists a covering $\mathcal{U}$ of $(X \times Y) - \{(p,q)\}$ such that $(X \times Y) - \{(p,q)\}$ is $\mathcal{U}$-covered with respect property (b) and $(X \times Y) - \{(p,q)\}$ is a $\mathcal{U}$-Maya space, then (p,q) does not make a hole in $X \times Y$.*

Proof. A consequence of Theorem 3.6 and Theorem 2.3 is the unicoherence of $(X \times Y) - \{(p,q)\}$, and so (p,q) does not make a hole in $X \times Y$.

For a smooth dendroid (X, v_X) and $p \in X$, set $\Gamma_p^X = \{x \in X : p \notin v_X x\} \cup \{p\}$, $\Omega_p^X = \{x \in X : p \in v_X x\}$ and if p satisfies that $\Omega_p^X - \{p\} \neq \emptyset$, then $\Delta_X(p)$ denotes the family of subsets of the form $S \cup \{p\}$ of X where S is an arc-component of $\Omega_p^X - \{p\}$.

Lemma 3.8. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and $(p,q) \in X \times Y$. Then each one of the following statements holds.*

(3.8.1) $g_X(\Gamma_p^X \times I) = \Gamma_p^X$.

(3.8.2) *The subset Γ_p^X of X is connected, $v_X \in \Gamma_p^X$ and Γ_p^X is contractible.*

(3.8.3) *If $q \neq v_Y$, then the set $(X \times \Gamma_q^Y) - \{(p,q)\}$ is contractible and so it has property (b).*

(3.8.4) *The set Ω_p^X is a subcontinuum of X and (Ω_p^X, p) is a smooth dendroid.*

(3.8.5) $g_{\Omega_p^X}(T \times I) = T$ *for each $T \in \Delta_X(p)$.*

(3.8.6) *Each element of $\Delta_X(p)$ has property (b).*

(3.8.7) *If* $T \in \Delta_X(v_X)$ *is such that* $y \notin T$ *and* $s \in I$, *then* $g_X(y, s) \in T$ *if only if* $s = 0$.

Proof. In order to show (3.8.1), let $(x, t) \in \Gamma_p^X \times I$ be arbitrary. Observe that the condition $x \in \Gamma_p^X$ implies that $v_X x \subseteq \Gamma_p^X$. Thus, (3.1.6) ensures that $g_X(x, t) \in \Gamma_p^X$. Then the inclusion $g_X(\Gamma_p^X \times I) \subseteq \Gamma_p^X$ holds. Now, in light of (3.1.5), we deduce that $\Gamma_p^X \subseteq g_X(\Gamma_p^X \times I)$.

The connectedness of Γ_p^X follows from facts that $v_X \in \bigcap\{v_X x : x \in \Gamma_p^X\}$ and $\Gamma_p^X = \bigcup\{v_X x : x \in \Gamma_p^X\}$. Now, the equality of (3.8.1) and the conditions (3.1.4) and (3.1.5) guarantee that $g_X|_{\Gamma_p^X \times I} : \Gamma_p^X \times I \to \Gamma_p^X$ is a contraction. Therefore, Γ_p^X is contractible. Then (3.8.2) is true.

We shall argue (3.8.3). Set $Z = (X \times \Gamma_q^Y) - \{(p, q)\}$. In order to get a contraction of A, define $G : A \times I \to A$ by $G((x, y), t) = (g_X(x, t), g_Y(y, t))$. First, let $((a, b), t) \in A \times I$ be arbitrary. By (3.8.1), we deduce that $G((a, b), t) \in X \times \Gamma_q^Y$. Now, we need to show that $G((a, b), t) \neq (p, q)$. To this end, suppose to the contrary that $G((a, b), t) = (p, q)$. Thus, $g_X(a, t) = p$ and $g_Y(b, t) = q$. Since $b \in \Gamma_q^Y$, we infer that $b = q$ and, by (3.1.5), we get $t = 1$. Hence, $a = g_X(a, 1) = p$. In conclusion, $(a, b) = (p, q)$, a contradiction. On the other hand, the continuity of G follows from (3.1.2). Finally, the conditions (3.1.4) and (3.1.5) guarantee that G is a contraction.

Observe that $p \in \bigcap\{px : x \in \Omega_p^X\}$ and $\Omega_p^X = \bigcup\{px : x \in \Omega_p^X\}$. Hence, Ω_p^X is connected. To show that Ω_p^X is closed in X, let $(\langle x_k \rangle, x) \in \mathbb{S}(X)$ be such that each $x_k \in \Omega_p^X$. Our assumption (X, v_X) is a smooth dendroid guarantees that $(\langle v_X x_k \rangle, v_X x) \in \mathbb{S}(C(X))$. Since $p \in v_X x_k$ for each $k \in \mathbb{N}$, we infer that $p \in v_X x$ and so $x \in \Omega_p^X$. This shows that Ω_p^X is closed in X. Therefore Ω_p^X is a subcontinuum of X. Hence, we conclude that (3.8.4) holds.

In order to prove (3.8.5), let $T \in \Delta_X(p)$ be arbitrary. First, we are going to argue the inclusion $g_{\Omega_p^X}(T \times I) \subseteq T$. Let $(x, t) \in T \times I$ be arbitrary. Notice that the condition $x \in T$ implies that $px \subseteq T$. Thus, by (3.1.6), we obtain that $g_{\Omega_p^X}(x, t) \in T$. Now, in light of (3.1.5), we deduce that $T \subseteq g_{\Omega_p^X}(T \times I)$.

A consequence of Proposition 2.2 and the fact that $g_{\Omega_p^X} : T \times I \to T$ is a contraction (see (3.8.5), (3.1.4) and (3.1.5)) is that T has property (b). So, (3.8.6) holds.

We are going to prove the first part of (3.8.7). Our assumption $y \notin T$ implies that $y \in X - \{v_X\}$ and $v_X y \cap T = \{v_X\}$. So, by (3.1.6), we have that $g_X(y,s) \in g_X(\{y\} \times I) \cap T = \{v_X\} = \{g_X(y,0)\}$. Applying (3.1.4), we infer that $s = 0$. The second part is immediate, if $s = 0$, then $g_X(y,0) = v_X \in T$.

Results below will be essential in the proof of the main theorems in the next section.

Lemma 3.9. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and let $(p,q) \in X \times Y$. If $T \in \Delta_X(p)$, then $T \times Y - \{(p,q)\}$ is $\{(T \times \{y\}) \cup (\{x\} \times Y) : (x,y) \in (T - \{p\}) \times (Y - \{q\})\}$-covered with respect property (b).*

Proof. Set $Z = (T \times Y) - \{(p,q)\}$, $E = T - \{p\}$ and $G = Y - \{q\}$. For each $(x,y) \in E \times G$, let $U(x,y) = (T \times \{y\}) \cup (\{x\} \times Y)$. Define $\mathcal{U} = \{U(x,y) : (x,y) \in E \times G\}$. Observe that $\mathcal{U}$ is a covering of Z.

First, by Corollary 3.2 and (3.8.6), Y and T have property (b). Thus, Theorem 3.3 guarantees that each element of $\mathcal{U}$ has property (b).

Next, fix $r \in G$. Set $M = T \times \{r\}$. Notice that M is a connected closed subset of Z having property (b). Also, $M \cap U(x,r) = M$ and $M \cap U(x,y) = \{(x,r)\}$ are connected for each $(x,y) \in E \times (G - \{r\})$.

Finally, let $x, w \in E$ and $y, z \in G$ be arbitrary. We have that $U(x,y) \cap U(w,z) \neq \emptyset$. Set $J = xw$. Since E is arcwise connected, we infer that $J \subseteq E$. Define $L(U(x,y), U(w,z)) = (J \times Y) \cup U(x,y)$. For sake of simplicity, L will represent to $L(U(x,y), U(w,z))$. By [9, (7.5)], we conclude that $J \times Y$ has property (b). Hence, since $(J \times Y) \cap U(x,y) = (J \times \{y\}) \cup (\{x\} \times Y))$ is connected, by Theorem 2.6, we obtain that L has property (b). Observe that $U(x,y) \cap U(w,z) \subseteq L$. We have that $U(w,z) \cap L = U(w,z)$ if $z = y$ and $U(w,z) \cap L = (J \times \{z\}) \cup (\{w\} \times Y)$ otherwise. Thus, the sets $L \cap M =$

M, $U(x,y) \cap L = U(x,y)$ and $U(w,z) \cap L$ are connected and the equality $(U(x,y) \cap M) \cup (U(w,z) \cap M) = M = L \cap M$ holds. Thus L fulfils all our requirements.

In conclusion, Z is $\mathcal{U}-$covered with respect to property (b).

Let X and Y be metric spaces. For a subset Z of $X \times Y$, the set of all elements $(\langle (x_k, y_k)\rangle, (x_0, y_0))$ of $\mathbb{S}(Z)$ such that each subsequence $\langle (x_{k_j}, y_{k_j})\rangle$ of $\langle (x_k, y_k)\rangle$ satisfies that the sets $\{x_{k_j} : j \in \mathbb{N}\}$ and $\{y_{k_j} : j \in \mathbb{N}\}$ are infinity will be represented by $\mathbb{S}^*(Z)$. This notation will be used for the rest of the paper.

Lemma 3.10. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and let $(p,q) \in X \times Y$. If $T \in \Delta_X(p)$, then $T \times Y - \{(p,q)\}$ has property (b).*

Proof. In light of Theorem 3.6, it suffices to show the existence of a covering $\mathcal{U}$ of $(T \times Y) - \{(p,q)\}$ such that $(T \times Y) - \{(p,q)\}$ is $\mathcal{U}$-covered with respect property (b) and $(T \times Y) - \{(p,q)\}$ is a $\mathcal{U}$-Maya space.

Set $Z = (T \times Y) - \{(p,q)\}$, $E = T - \{p\}$ and $G = Y - \{q\}$. For each $(x,y) \in E \times G$, let $U(x,y) = (T \times \{y\}) \cup (\{x\} \times Y)$. Define $\mathcal{U} = \{U(x,y) : (x,y) \in E \times G\}$. Observe that $\mathcal{U}$ is a covering of Z. Lemma 3.9 guarantees that Z is $\mathcal{U}$-covered with respect property (b).

In order to prove that Z is a $\mathcal{U}$-Maya space, let $(\langle (x_k, y_k)\rangle, (x_0, y_0)) \in \mathbb{S}(Z)$ be arbitrary. Taking subsequences, if it is necessary, by Lemma 3.4, we may assume that $(\langle (x_k, y_k)\rangle, (x_0, y_0)) \in \mathbb{S}^*(Z)$ and we only need to consider the following cases.

Case I. $\{x_k : k \in \mathbb{N} \cup \{0\}\} \subseteq E$.

Fix $w \in G$. Consider $\mathcal{V} = \{U(x_k, w) : k \in \mathbb{N} \cup \{0\}\}$. Observe that $\{(x_k, y_k) : k \in \mathbb{N} \cup \{0\}\} \subseteq \bigcup \mathcal{V}$ and $(p, w) \in \bigcap \mathcal{V}$.

Define $\varphi : F_H \to Z$ by

$$\varphi(t,u) = \begin{cases} (g_{\Omega_p^X}(x_l, 3t), w) & \text{if } (t,u) \in J_l \text{ and } t \leq \frac{1}{3} \\ (x_l, g_Y(w, 2-3t)) & \text{if } (t,u) \in J_l \text{ and } \frac{1}{3} \leq t \leq \frac{2}{3} \\ (x_l, g_Y(y_l, 3t-2)) & \text{if } (t,u) \in J_l \text{ and } \frac{2}{3} \leq t \end{cases}$$

Let us show that φ is monotone with respect to $\mathcal{V}$. Let $k \in \mathbb{N} \cup \{0\}$ be arbitrary. In order to prove that $\varphi^{-1}(U(x_k, w))$ is connected, define $A = \{l \in \mathbb{N} \cup \{0\} : x_k = x_l\}$ and $B = \{l \in \mathbb{N} \cup \{0\} : x_k \neq x_l\}$. We shall prove the following claims.

Claim 1. $\bigcup_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{3}) \subseteq \varphi^{-1}(U(x_k, w))$.

If $(t, u) \in \bigcup_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{3})$, by (3.8.5), then $\varphi(t, u) \in T \times \{w\} \subseteq U(x_k, w)$.

Claim 2. $\bigcup_{l \in A} J_l \subseteq \varphi^{-1}(U(x_k, w))$.

If $(t, u) \in \bigcup_{l \in A} J_l$ and $t \geq \frac{1}{3}$, then $\varphi(t, u) \in \{x_k\} \times Y \subseteq U(x_k, w)$. From this and Claim 1, we can conclude that J_l is a subset of $\varphi^{-1}(U(x_k, w))$ for each $l \in A$.

Claim 3. $J_l \cap \varphi^{-1}(U(x_k, w)) = J_l(\frac{1}{3})$ for each $l \in B$.

Let $l \in B$ be arbitrary. Claim 1 guarantees that $J_l(\frac{1}{3}) \subseteq \varphi^{-1}(U(x_k, w))$. Now, from the definition of φ, the inclusion $(t, u) \in J_l \cap \varphi^{-1}(U(x_k, w))$ and the inequality $x_k \neq x_l$ imply that $t \leq \frac{1}{3}$. Thus, $J_l \cap \varphi^{-1}(U(x_k, w))$ is a subset of $J_l(\frac{1}{3})$.

Next, invoke our last claims to show that $\varphi^{-1}(U(x_k, w)) = \left(\bigcup_{l \in A} J_l \right) \cup \left(\bigcup_{l \in B} J_l(\frac{1}{3}) \right)$ is connected.

Finally, notice that $(0, 0) \in \varphi^{-1}(\bigcap \mathcal{V})$ and $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$. Thus, $\mathcal{V}$, F_H, φ and $\langle (1, \frac{1}{k}), (1, 0) \rangle \in \mathbb{S}(F_H)$ fulfil all our requirements.

Case II. $\{x_k : k \in \mathbb{N}\} \subseteq E$ and $x_0 = p$.

Then $y_0 \neq q$. So, we may assume that $\{y_k : k \in \mathbb{N}\} \subseteq G$. Fix $z \in E$ and consider $\mathcal{V} = \{U(z, y_k) : k \in \mathbb{N}\}$. Then $(z, v_Y) \in \bigcap \mathcal{V}$. Let $\varphi : F_H \to Z$ be define by

$$\varphi(t, u) = \begin{cases} (z, g_Y(y_l, 3t)), & \text{if } (t, u) \in J_l \text{ and } t \leq \frac{1}{3} \\ (g_{\Omega_p^X}(z, 2 - 3t), y_l), & \text{if } (t, u) \in J_l \text{ and } \frac{1}{3} \leq t \leq \frac{2}{3} \\ (g_{\Omega_p^X}(x_l, 3t - 2), y_l), & \text{if } (t, u) \in J_l \text{ and } \frac{2}{3} \leq t \end{cases}$$

In order to prove that φ is monotone with respect to $\mathcal{V}$, let $k \in \mathbb{N} \cup \{0\}$ be arbitrary and, set $A = \{l \in \mathbb{N} \cup \{0\} : y_l = y_k\}$ and $B = \{l \in \mathbb{N} \cup \{0\} : y_l \neq y_k\}$. The following claims will give that $\varphi^{-1}(U(z, y_k))$ is connected.

Claim 1. $\bigcup_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{3}) \subseteq \varphi^{-1}(U(z, y_k))$.

If $(t, u) \in \bigcup_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{3})$, then $\varphi(t, u) \in \{z\} \times Y \subseteq U(z, y_k)$. This guarantees the inclusion $J_l(\frac{1}{3}) \subseteq \varphi^{-1}(U(z, y_k)$ for each $l \in \mathbb{N} \cup \{0\}$.

Claim 2. $\bigcup_{l \in A} J_l \subseteq \varphi^{-1}(U(z, y_k))$.

Let $(t, u) \in \bigcup_{l \in A} J_l$ be arbitrary. In light of Claim 1, we only need to suppose that $t \geq \frac{1}{3}$. By (3.8.5), we have that $\varphi(t, u) \in T \times \{y_k\} \subseteq U(z, y_k)$.

Claim 3. $J_l(\frac{1}{3}) = J_l \cap \varphi^{-1}(U(z, y_k))$ for each $l \in B$.

Let $l \in B$ be arbitrary. Claim 1 ensures that $J_l(\frac{1}{3}) \subseteq J_l \cap \varphi^{-1}(U(z, y_k))$. Now, if $(t, u) \in J_l$ is such that $\varphi(t, u) \in U(z, y_k)$, since $y_k \neq y_l$, then $\varphi(t, u) \in \{z\} \times Y$ and $t \leq \frac{1}{3}$. The proof of our claim is complete.

Thus, claims 1, 2 and 3 imply that $\varphi^{-1}(U(z, y_k)) = \left(\bigcup_{l \in A} J_l\right) \cup \left(\bigcup_{l \in B} J_l(\frac{1}{3})\right)$ is connected.

On the other hand, we have $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for each $k \in \mathbb{N}$ and $(0, 0) \in \varphi^{-1}(\bigcap \mathcal{V})$. So, $\mathcal{V}$, F_H, φ and $\langle (1, \frac{1}{k}), (1, 0) \rangle \in \mathbb{S}(F_H)$ satisfies all our requirements.

In conclusion, Z is a $\mathcal{U}$-Maya space.

Lemma 3.11. *Let (X, v_X) and (Y, v_Y) be smooth dendroid and let $(p, q) \in X \times Y$. If $z \in v_X p - \{v_X, p\}$ and $T \in \Delta_Y(q)$, then $(\{z\} \times Y) \cup ((X \times T) - \{(p, q)\})$ has property (b).*

Proof. In light Theorem 3.6, we need to prove that there exists a covering $\mathcal{U}$ of $(\{z\} \times Y) \cup ((X \times T) - \{(p, q)\})$ such that $(\{z\} \times Y) \cup ((X \times T) - \{(p, q)\})$ is $\mathcal{U}$-covered with respect property (b) and $(\{z\} \times Y) \cup ((X \times T) - \{(p, q)\})$ is a $\mathcal{U}$-Maya space.

Set $Z = (\{z\} \times Y) \cup ((X \times T) - \{(p,q)\})$, $E = \{z\} \times Y$ and $G = (X \times T) - \{(p,q)\}$. Consider $\mathcal{U} = \{E, G\}$. Notice that $\mathcal{U}$ is a covering of Z and $\bigcap \mathcal{U} \neq \emptyset$.

Let us argue that Z is $\mathcal{U}-$covered with respect to property (b).

By Corollary 3.2 and Lemma 3.10, we conclude that each element of $\mathcal{U}$ has property (b). Now, set $M = E = L(E,G)$. We have that M and L are connected closed subsets of Z having property (b). The sets $M \cap E = E$ and $M \cap G = \{z\} \times T$ are connected. Thus M satisfies the required properties of our definition. For sake of simplicity, L will represent to $L(E,G)$. Observe that the inclusions $E \cap G = \{z\} \times T \subseteq L$ and $(E \cap M) \cup (G \cap M) \subseteq L \cap M$ hold and the sets $L \cap E = E$, $L \cap G = \{z\} \times T$ and $L \cap M = M$ are connected and non-empty. Thus, L fulfilling the conditions in the definition. We can conclude that Z is $\mathcal{U}-$covered with respect to property (b).

Now, in order to prove that Z is a $\mathcal{U}-$Maya space, let $(\langle (x_k, y_k) \rangle, (x_0, y_0)) \in \mathbb{S}(Z)$ be arbitrary. Taking subsequences, if it is necessary, by Lemma 3.4 and since E is a closed subset of Z, we only need to assume that each $(x_k, y_k) \in G - E$, $(x_0, y_0) \in E - G$ and $(\langle (x_k, y_k) \rangle, (x_0, y_0)) \in \mathbb{S}^*(Z)$.

The assumptions $(x_0, y_0) \in E - G$ and each $(x_k, y_k) \in G - E$ imply that $x_0 = z$, $y_0 \in \Omega_q^Y - T$ and $x_k \neq z$ for each $k \in \mathbb{N}$. Hence, we may assume that $p \notin \{x_k : k \in \mathbb{N}\}$ and $q \notin \{y_k : k \in \mathbb{N}\}$. We will consider two cases:

Case I. $z \in v_X x_k$ for each $k \in \mathbb{N}$.

In light of (3.8.4), we may consider the mappings $g_{\Omega_q^Y}$ and $g_{\Omega_z^X}$. Let $\varphi : F_H \to Z$ be defined by

$$\varphi(t,u) = \begin{cases} (z, g_{\Omega_q^Y}(y_l, 2t)), & \text{if } (t,u) \in J_l \text{ and } t \leq \frac{1}{2} \\ (g_{\Omega_z^X}(x_l, 2t-1), y_l), & \text{if } (t,u) \in J_l \text{ and } \frac{1}{2} \leq t \end{cases}$$

Notice that φ is well defined, the continuity of φ follows from (3.1.1), $\varphi(0,0) \in \bigcap \mathcal{U}$ and so $\varphi^{-1}(\bigcap \mathcal{U}) \neq \emptyset$.

The connectedness of $\varphi^{-1}(E)$ and $\varphi^{-1}(G)$ shall be a consequence of the below claims.

Claim 1. $\bigcup_{l\in\mathbb{N}} J_l(\frac{1}{2}) \subseteq \varphi^{-1}(G) \cap \varphi^{-1}(E)$.

Let $l \in \mathbb{N}$ be arbitrary and let $(t,u) \in J_l$ be such that $t \leq \frac{1}{2}$. Then $\varphi(t,u) \in E$ and, by (3.8.5), $\varphi(t,u) \in G$. This implies that $(t,u) \in \varphi^{-1}(G) \cap \varphi^{-1}(E)$.

Claim 2. $J_0 \subseteq E$.

Notice that $\varphi(J_0(\frac{1}{2})) \subseteq E$ and, by the definition of $g_{\Omega_z^X}$ and our assumption $z = x_0$, we obtain that $\varphi(t,0) \in E$ for all $t \in [\frac{1}{2}, 1]$. In conclusion, $\varphi(J_0) \subseteq E$.

Claim 3. $J_l \cap \varphi^{-1}(E) = J_l(\frac{1}{2})$ for each $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$ be arbitrary. First, from the fact that $x_l \neq z$, by (3.1.4), for each $(t,u) \in J_l$ such that $\varphi(t,u) \in E$, we have that $t \leq \frac{1}{2}$. This implies that $J_l \cap \varphi^{-1}(E)$ is a subset of $J_l(\frac{1}{2})$. The inclusion $J_l(\frac{1}{2}) \subseteq J_l \cap \varphi^{-1}(E)$ is guaranteed by Claim 1.

Claim 4. $\bigcup_{l\in\mathbb{N}} J_l \subseteq \varphi^{-1}(G)$.

Let $l \in \mathbb{N}$ be arbitrary. Claim 1 ensures that $J_l(\frac{1}{2}) \subseteq \varphi^{-1}(G)$. Next, if $(t,u) \in J_l$ satisfies that $t \geq \frac{1}{2}$, from the fact that $y_k \in T$, by (3.8.5), we infer that $\varphi(t,u) \in G$. Therefore, $J_l \subseteq \varphi^{-1}(G)$ for each $l \in \mathbb{N}$.

Claim 5. $J_0 \cap \varphi^{-1}(G) = \{(0,0)\}$.

By (3.8.7) and from our assumption $y_0 \notin T$, we infer that if $\varphi(y_0, t) \in G$, then $t = 0$. This proves our claim.

Thus, from claims 1-5, it follows that $\varphi^{-1}(E) = J_0 \cup \bigcup_{l\in\mathbb{N}} J_l(\frac{1}{2})$ and $\varphi^{-1}(G) = \bigcup_{l\in\mathbb{N}} J_l$ are connected. This implies that φ is monotone with respect to $\mathcal{U}$.

Observe that $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$. In conclusion $\mathcal{U}$, F_H, φ and $\langle(1, \frac{1}{k}), (1,0)\rangle \in \mathbb{S}(F_H)$ fulfil all our requirement.

Case II. $z \notin v_X x_k$ for each $k \in \mathbb{N}$.

Our assumption and the facts that $z \in v_X p - \{p\}$ and $x_0 = z$ imply that $p \notin v_X x_k$ for each $k \in \mathbb{N} \cup \{0\}$.

Define $\varphi : F_H \to Z$ by

$$\varphi(t,u) = \begin{cases} (g_X(x_l, 2t), q), & \text{if } (t,u) \in J_l \text{ and } t \le \frac{1}{2}, \\ (x_l, g_{\Omega_q^X}(y_l, 2t-1)), & \text{if } (t,u) \in J_l \text{ and } t \ge \frac{1}{2} \end{cases}$$

to get a map. Notice that $\varphi(\frac{1}{2}, 0) = (z, q) \in \bigcap \mathcal{U}$ and hence $\varphi^{-1}(\bigcap \mathcal{U}) \neq \emptyset$.

Next, we are going to show that $\varphi^{-1}(E)$ and $\varphi^{-1}(G)$ are connected. To this end, we prove the following claims.

Claim 1. $J_0 \cap \varphi^{-1}(E) = \{(t,0) \in J_0 : t \ge \frac{1}{2}\}$.

First, let $t \in I$ be such that $\varphi(t,0) \in E$. Since $x_0 = z$, by (3.1.5), we deduce that $t \ge \frac{1}{2}$. This implies that $J_0 \cap \varphi^{-1}(E) \subseteq \{(t,0) \in J_0 : t \ge \frac{1}{2}\}$. Now, if $t \in [\frac{1}{2}, 1]$, the equality $x_0 = z$ and (3.8.5) guarantee that $\varphi(t,0) \in E$. The conclusion is that $\{(t,0) \in J_0 : t \ge \frac{1}{2}\}$ is a subset of $\varphi^{-1}(E)$. This proves our claim.

Claim 2. $\varphi^{-1}(E) \cap \bigcup\limits_{l \in \mathbb{N}} J_l = \emptyset$.

This claim follows from the fact that $z \notin g_X(\{x_l\} \times I)$ for each $l \in \mathbb{N}$ (see (3.1.6)).

Claim 3. $\bigcup\limits_{l \in \mathbb{N}} J_l \subseteq \varphi^{-1}(G)$.

By (3.8.5) and $y_k \in T$, we have that $\varphi(J_l) \subseteq G$ for each $l \in \mathbb{N}$.

Claim 4. $J_0 \cap \varphi^{-1}(G) = J_0(\frac{1}{2})$.

The inclusion $q \in T$ guarantees that $\varphi(J(\frac{1}{2})) \subseteq G$. On the other hand, if $t \in I$ is such that $\varphi(t,0) \in G$, by (3.8.7), we obtain that $t \le \frac{1}{2}$. The proof of this claim is finished.

Thus, by claim 1-4, we obtain that $\varphi^{-1}(E) = \{(t,0) \in J_0 : t \ge \frac{1}{2}\}$ and $\varphi^{-1}(G) = J_0(\frac{1}{2}) \cup \bigcup\limits_{l \in \mathbb{N}} J_l$ are connected. This implies that φ is monotone with respect to $\mathcal{U}$.

Finally, notice that $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$. Therefore, $\mathcal{U}$, F_H, φ and $\langle (1, \frac{1}{k}), (1,0) \rangle \in \mathbb{S}(F)$ fulfil all our requirements.

We have that Z is a $\mathcal{U}-$Maya space.

Lemma 3.12. *Let (X, v_X) and (Y, v_Y) be smooth dendroid and let $(p, q) \in X \times Y$. If $z \in v_X p - \{v_X, p\}$, then $((X \times \Gamma_q^Y) - \{(p, q)\}) \cup (\{z\} \times Y)$ has property (b).*

Proof. For sake of simplicity, set $Z = ((X \times \Gamma_q^Y) - \{(p, q)\}) \cup (\{z\} \times Y)$. In light of Theorem 3.6, if suffices to prove that exists a covering $\mathcal{U}$ of Z such that Z is $\mathcal{U}-$covered with respect to property (b) and Z is a $\mathcal{U}-$Maya space.

First, set $E = (X \times \Gamma_q^Y) - \{(p, q)\}$ and $G = \{z\} \times Y$. Consider $\mathcal{U} = \{E, G\}$. Notice that $\mathcal{U}$ is a covering of Z and $\bigcap \mathcal{U} \neq \emptyset$. Second, Corollary 3.2 and (3.8.3) guarantees that each element of $\mathcal{U}$ has property (b). Now, set $M = G = L(E, G)$. Then M and L are connected closed subsets of Z having property (b). Observe that $M \cap G = G$ and $M \cap E = \{z\} \times \Gamma_q^Y$ are connected. The symbol L will represent to $L(E, G)$. Notice that L is a connected closed subset of Z having property (b). These sets satisfy: $E \cap G = \{z\} \times \Gamma_q^Y \subseteq L$, $L \cap G = G$, $L \cap E = \{z\} \times \Gamma_q^Y$ are connected, $L \cap M = M \neq \emptyset$ and $(G \cap M) \cup (E \cap M) \subseteq L \cap M$. Thus, L fulfilling the conditions in the definition. This finishes the proof of that Z is $\mathcal{U}-$covered with respect to property (b).

In order to prove that Z is $\mathcal{U}-$Maya space, let $(\langle (x_k, y_k) \rangle, (x_0, y_0)) \in \mathbb{S}(Z)$ be arbitrary. Taking subsequences, if it is necessary, by Lemma 3.4 and the condition G is a closed subset of Z, we only need to assume that $\{(x_k, y_k) : k \in \mathbb{N}\} \subseteq E - G$, $(x_0, y_0) \in G - E$ and $(\langle (x_k, y_k) \rangle, (x_0, y_0)) \in \mathbb{S}^*(Z)$.

The assumptions $(x_0, y_0) \in G - E$ and each $(x_k, y_k) \in E - G$ guarantee that $x_0 = z$, $y_0 \in \Omega_q^Y - \{q\}$ and $z \notin \{x_k : k \in \mathbb{N}\}$. Thus, we may assume that $x_k \neq p$ and $y_k \neq q$ for each $k \in \mathbb{N}$.

Now, we consider two cases.

Case I. $z \in v_X x_k$ for each $k \in \mathbb{N}$.

By (3.8.5), we can consider the mapping $g_{\Omega_z^X}$. Let $\varphi : F_H \to Z$ be defined by

$$\varphi(t, u) = \begin{cases} (z, g_Y(y_l, 2t)), & \text{if } (t, u) \in J_l \text{ and } t \leq \frac{1}{2}, \\ (g_{\Omega_z^X}(x_l, 2t - 1), y_l), & \text{if } (t, u) \in J_l \text{ and } t \geq \frac{1}{2}. \end{cases}$$

Observe that φ is a map and $(0,0) \in \varphi^{-1}(\bigcap \mathcal{U})$.

Now, we shall prove the claims below to argue the connectedness of $\varphi^{-1}(E)$ and $\varphi^{-1}(G)$.

Claim 1. $\bigcup_{l\in\mathbb{N}} J_l(\frac{1}{2}) \subseteq \varphi^{-1}(E) \cap \varphi^{-1}(G)$.

From the definition of φ, it follows that $\varphi(J_l(\frac{1}{2})) \subseteq G$. Now, if $l \in \mathbb{N}$, the inclusion $y_l \in \Gamma_q^Y$ and (3.8.1) guarantee that $\varphi(t,u) \in E$ for each $(t,u) \in J_l(\frac{1}{2})$.

Claim 2. $\bigcup_{l\in\mathbb{N}} J_l \subseteq \varphi^{-1}(E)$.

If $(t,u) \in \bigcup_{l\in\mathbb{N}} J_l$ and $t \geq \frac{1}{2}$, since $y_l \in \Gamma_q^Y$, then $\varphi(t,u) \in E$. This and Claim 1 prove that $\varphi(J_l)$ is contained in E for each $l \in \mathbb{N}$.

Claim 3. $J_0(e) = J_0 \cap \varphi^{-1}(E)$ where $e < \frac{1}{2}$ is such that $g_Y(y_0, 2e) = q$.

By (3.1.6), we deduce that $\varphi(J_0(e)) \subseteq \{z\} \times \Gamma_q^Y \subseteq E$. Then $J_0(e) \subseteq J_0 \cap \varphi^{-1}(E)$. Now, let $t \in I$ such that $\varphi(t,0) \in E$. From the fact that $y_0 \in \Omega_q^Y - \{q\}$, it follows that $t \leq \frac{1}{2}$. Hence, $g_Y(y_0, 2t) \in \Gamma_q^Y$. This implies that $g_Y(y_0, 2t) \in v_Y q = g_Y(\{y_0\} \times [0, 2e])$ and so $t \leq e$ (see (3.1.3)). We conclude that $J_0 \cap \varphi^{-1}(E)$ is a subset of $J_0(e)$.

Claim 4. $\varphi^{-1}(G) \cap J_l = J_l(\frac{1}{2})$ for each $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$ be arbitrary. Claim 1 ensures that $J_l(\frac{1}{2})$ is contained in $\varphi^{-1}(G)$. Next, let $(t,u) \in J_l$ be such that $\varphi(t,u) \in G$. Since $z \neq x_l$, by (3.1.4), we have that $t \leq \frac{1}{2}$. This proves our claim.

Claim 5. $J_0 \subseteq \varphi^{-1}(G)$.

By the definition of $g_{\Omega_z^X}$, it follows that $\varphi(t,0) \in G$ for all $t \in I$.

Thus, from claims 2-5, it follows that $\varphi^{-1}(E) = J_0(e) \cup \bigcup_{l\in\mathbb{N}} J_l$ and $\varphi^{-1}(G) = J_0 \cup \bigcup_{l\in\mathbb{N}} J_l(\frac{1}{2})$ are connected. This proves that φ is monotone with respect to $\mathcal{U}$.

Notice that $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$. So, $\mathcal{U}$, F_H, φ and $\langle(1, \frac{1}{k}), (1,0)\rangle \in \mathbb{S}(F_H)$ satisfy the required properties.

Case II. $z \notin v_X x_k$ for each $k \in \mathbb{N}$.

Our assumption and the choice $z \in v_X p - \{v_X, p\}$ imply that $p \notin v_X x_k$ for each $k \in \mathbb{N} \cup \{0\}$.

Define $\varphi : F_H \to Z$ by

$$\varphi(t,u) = \begin{cases} (g_X(x_l, 2t), v_Y), & \text{if } (t,u) \in J_l \text{ and } t \leq \frac{1}{2}, \\ (x_l, g_Y(y_l, 2t-1)), & \text{if } (t,u) \in J_l \text{ and } t \geq \frac{1}{2}, \end{cases}$$

to get a map. Notice that $(0,0) \in \varphi^{-1}(\bigcap \mathcal{U})$.

Next, let us argue that φ is monotone with respect to $\mathcal{U}$. To this end, we are going to prove the following claims.

Claim 1. $\bigcup_{l \in \mathbb{N}} J_l \subseteq \varphi^{-1}(E)$.

By the definition of φ, we deduce that $\varphi(J_l(\frac{1}{2})) \subseteq X \times \{v_Y\} \subseteq E$ for each $l \in \mathbb{N}$. Now, if $(t,u) \in \bigcup_{l \in \mathbb{N}} J_l$ is such that $t \geq \frac{1}{2}$, by (3.8.1), $y_k \in \Gamma_q^Y$ and $x_l \neq p$, we have that $\varphi(t,u) \in \{x_l\} \times \Gamma_q^Y \subseteq E$.

Claim 2. $J_0(e) = J_0 \cap \varphi^{-1}(E)$ where $e \in [\frac{1}{2}, 1]$ is such that $g_Y(y_0, 2e-1) = q$.

First, notice that $\varphi(J_0(\frac{1}{2})) \subseteq X \times \{v_Y\} \subseteq E$. Second, by (3.1.6), if $t \in [\frac{1}{2}, e]$, then $\varphi(t,u) \in \{z\} \times v_Y q \subseteq E$. This proves that $J_0(e)$ is a subset of $\varphi^{-1}(E)$. Now, let $t \in I$ be such that $\varphi(t,0) \in E$. Assume that $t \geq \frac{1}{2}$. By (3.1.6), then $\varphi(t,0) \in \{z\} \times g_Y(\{y_0\} \times [0, 2e-1])$. Hence, in light of (3.1.3), we deduce that $t \leq e$. In conclusion, $J_0 \cap \varphi^{-1}(E) \subseteq J_0(e)$.

Claim 3. $J_l \cap \varphi^{-1}(G) = \emptyset$ for each $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$ be arbitrary. From the fact that $z \notin g_X(\{x_l\} \times I)$, we deduce that $\varphi(J_l) \cap G = \emptyset$ (see (3.1.6)). This shows our claim.

Claim 4. $J_0 \cap \varphi^{-1}(G) = \{(t,0) \in J_0 : t \geq \frac{1}{2}\}$.

If $t \in [\frac{1}{2}, 1]$, since $x_0 = z$, we obtain that $\varphi(t,u) \in G$. Hence, $\{(t,0) \in J_0 : t \geq \frac{1}{2}\}$ is a subset of $\varphi^{-1}(G)$. Now, let $t \in I$ be such that $\varphi(t,0) \in G$. By (3.1.5), we deduce that $t \geq \frac{1}{2}$. This finishes the proof of our claim.

So, invoke claims 1-4 to prove that $\varphi^{-1}(E) = J_0(e) \cup \bigcup_{l\in\mathbb{N}} J_l$ and $\varphi^{-1}(G) = \{(t,0) \in J_0 : t \geq \frac{1}{2}\}$ are connected. This implies that φ is monotone with respect to $\mathcal{U}$.

Finally, we have $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$. Hence, $\mathcal{U}$, F_H, φ and $\langle(1, \frac{1}{k}), (1,0)\rangle \in \mathbb{S}(F_H)$ fulfil all our requirements.

Therefore, Z is a $\mathcal{U}-$Maya space.

Lemma 3.13. *Let (X, v_X) and (Y, v_Y) be smooth dendroids, let $(p,q) \in X \times Y$. If $p \in Ncut(X) - E(X)$, $q \in Y - E(Y)$, $z \in v_X p - \{v_X, p\}$ and $T \in \Delta_Y(q)$, then $((X \times (\Gamma_q^Y \cup T)) - \{(p,q)\}) \cup (\{z\} \times Y)$ has property (b).*

Proof. For sake of simplicity denote $((X \times (\Gamma_q^Y \cup T)) - \{(p,q)\}) \cup (\{z\} \times Y)$ by Z. To show that Z has property (b), by Theorem 3.6, it suffices to verify that there exists a covering $\mathcal{U}$ of Z such that Z is $\mathcal{U}-$covered with respect to property (b) and Z is a $\mathcal{U}-$Maya space.

In order to define $\mathcal{U}$, set $E = ((X \times \Gamma_q^Y) - \{(p,q)\}) \cup (\{z\} \times Y)$ and $G = ((X \times T) - \{(p,q)\}) \cup (\{z\} \times Y)$. Consider $\mathcal{U} = \{E, G\}$. Observe that $\mathcal{U}$ is a covering of Z and $\bigcap \mathcal{U} \neq \emptyset$. Next, we are going to show that Z is $\mathcal{U}-$covered with respect to property (b).

Notice that E and G has property (b) by Lemma 3.11 and Lemma 3.12. Thus, each element of $\mathcal{U}$ has property (b).

Now, set $M = \{z\} \times Y$. We have that M is a connected closed subset of Z having property (b). Notice that $M \cap E = M = M \cap G$ are connected. On other hand, from the fact that $p \in Ncut(X) - E(X)$, we have that $X \times \{q\} - \{(p,q)\}$ is connected. Hence, the equality $E \cap G = M \cup ((X \times \{q\}) - \{(p,q)\})$ shows that $E \cap G$ is connected. Now, take $L(E,G) = G$. Then $L(E,G)$ has property (b), the sets $L(E,G) \cap E = E \cap G$, $L(E,G) \cap G = G$ and $L(E,G) \cap M = M$ are connected, the inclusion $E \cap G \subseteq L(E,G)$ holds and $(M \cap E) \cup (M \cap G) = M \subseteq L(E,G) \cap M = M$. This finishes the proof that Z is $\mathcal{U}$-covered with respect to property (b).

In order to prove that Z is $\mathcal{U}$-Maya space, let $(\langle(x_k, y_k)\rangle, (x_0, y_0)) \in \mathbb{S}(Z)$ be arbitrary. Taking subsequences, if it is necessary, by Lemma 3.4 and the condition G is a closed subset of Z, we only need to assume that $\{(x_k, y_k) : k \in \mathbb{N}\} \subseteq E - G$, $(x_0, y_0) \in G - E$ and $(\langle(x_k, y_k)\rangle, (x_0, y_0)) \in \mathbb{S}^*(Z)$.

Since $(x_0, y_0) \in G - E$, we obtain that $x_0 \neq z$ and $y_0 \in T - \{q\}$. Thus, we may suppose that $\{x_k : k \in \mathbb{N}\} \subseteq X - \{z\}$ and $\{y_k : k \in \mathbb{N}\} \subseteq \Gamma_q^Y - \{q\}$.

Taking subsequences, if it is necessary, we consider the following cases.

Case I. $z \in v_X x_l$ for each $l \in \mathbb{N}$.

In light of (3.8.4), we can consider the mapping $g_{\Omega_z^X}$. Let $\varphi : F_H \to Z$ be defined by

$$\varphi(t,u) = \begin{cases} (z, g_Y(y_l, 2t)), & \text{if } (t,u) \in J_l \text{ and } t \leq \frac{1}{2}, \\ (g_{\Omega_z^X}(x_l, 2t-1), y_l), & \text{if } (t,u) \in J_l \text{ and } \frac{1}{2} \leq t. \end{cases}$$

Now, we are going to prove that φ is monotone with respect to $\mathcal{U}$. To this end, we shall show the following claims.

Claim 1. $\bigcup\limits_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{2}) \subseteq \varphi^{-1}(E) \cap \varphi^{-1}(G)$.

If $(t,u) \in \bigcup\limits_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{2})$, then $\varphi(t,u) \in \{z\} \times Y \subseteq E \cap G$. Hence, $J_l(\frac{1}{2}) \subseteq \varphi^{-1}(E) \cap \varphi^{-1}(G)$ for each $l \in \mathbb{N} \cup \{0\}$.

Claim 2. $J_0 \subseteq \varphi^{-1}(G)$.

Let $(t,u) \in J_0$ be such that $\frac{1}{2} \leq t$. Since $y_0 \in T$, by (3.8.5), we deduce that $\varphi(t,u) \in (X \times T) - \{(p,q)\} \subseteq G$. This and Claim 1 imply that $J_0 \subseteq \varphi^{-1}(G)$.

Claim 3. $J_l(\frac{1}{2}) = J_l \cap \varphi^{-1}(G)$ for each $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$ be arbitrary. The inclusion $J_l(\frac{1}{2}) \subseteq J_l \cap \varphi^{-1}(G)$ is guaranteed by Claim 1. Since $y_l \notin T$, if $(t,u) \in J_l$ is such that $\varphi(t,u) \in G$, then $\varphi(t,u) \in \{z\} \times Y$ (see (3.8.5)) and, by (3.1.4) and $x_l \neq z$, we obtain that $t \leq \frac{1}{2}$. This shows that $J_l \cap \varphi^{-1}(G) \subseteq J_l(\frac{1}{2})$ for each $l \in \mathbb{N}$.

Claim 4. $J_0(\frac{1}{2}) = J_0 \cap \varphi^{-1}(E)$.

In light of Claim 1, we only need to prove that $J_0 \cap \varphi^{-1}(E)$ is a subset of $J_0(\frac{1}{2})$. Let $t \in I$ be such that $\varphi(t,0) \in E$. If t were greater than $\frac{1}{2}$, since $x_0 \neq z$, by (3.1.4), $\varphi(t,0)$ would be an element of $(X \times \Gamma_q^Y) - \{(p,q)\}$ and this would imply that $y_0 \in \Gamma_q^Y$, a contradiction. We conclude that $(t,0) \in J_0(\frac{1}{2})$.

Claim 5. $\bigcup\limits_{l \in \mathbb{N}} J_l \subseteq \varphi^{-1}(E)$.

If $(t,u) \in \bigcup\limits_{l \in \mathbb{N}} J_l$ is such that $t \geq \frac{1}{2}$, by (3.8.1), we have that $\varphi(t,u) \in (X \times \Gamma_q^Y) - \{(p,q)\}$. This and Claim 1 prove that each J_l is contained in $\varphi^{-1}(E)$.

Thus, by claims 1-5, we obtain that $\varphi^{-1}(E) = J_0(\frac{1}{2}) \cup \bigcup\limits_{l \in \mathbb{N}} J_l$ and $\varphi^{-1}(G) = J_0 \cup \bigcup\limits_{l \in \mathbb{N}} J_l(\frac{1}{2})$ are connected. We conclude that φ is monotone with respect to $\mathcal{U}$.

Observe that $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$ and $(0,0) \in \varphi^{-1}(\bigcap \mathcal{U})$. Therefore, $\mathcal{U}$, F_H, φ and $\langle (1, \frac{1}{k}), (1,0) \rangle \in \mathbb{S}(F_H)$ fulfil all our requirements.

Case II. $z \notin v_X x_l$ for each $l \in \mathbb{N}$.

Define $\varphi : F_H \to Z$ by

$$\varphi(t,u) = \begin{cases} (z, g_Y(y_l, 3t)), & \text{if } (t,u) \in J_l \text{ and } t \leq \frac{1}{3}, \\ (g_X(z, 2-3t), y_l), & \text{if } (t,u) \in J_l \text{ and } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ (g_X(x_l, 3t-2), y_l), & \text{if } (t,u) \in J_l \text{ and } \frac{2}{3} \leq t. \end{cases}$$

Next, let us show the connectedness of $\varphi^{-1}(E)$ and $\varphi^{-1}(G)$.

Claim 1. $\bigcup\limits_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{3}) \subseteq \varphi^{-1}(E) \cap \varphi^{-1}(G)$.

If $(t,u) \in \bigcup\limits_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{3})$, then $\varphi(t,u) \in \{z\} \times Y \subseteq E \cap G$. Hence, we obtain that $J_l(\frac{1}{3}) \subseteq \varphi^{-1}(E \cap G) = \varphi^{-1}(E) \cap \varphi^{-1}(G)$ for each $l \in \mathbb{N} \cup \{0\}$.

Claim 2. $\bigcup\limits_{l \in \mathbb{N}} J_l \subseteq \varphi^{-1}(E)$.

Let t be arbitrary. Claim 1 ensures that $J_l(\frac{1}{3}) \subseteq \varphi^{-1}(E)$ for each $l \in \mathbb{N}$. Now, let $(t,u) \in J_l$ be such that $t \geq \frac{1}{3}$. Then, since $y_l \in \Gamma_q^Y$, we deduce that $\varphi(t,u) \in (X \times \Gamma_q^Y) - \{(p,q)\} \subseteq E$. Thus, $\varphi^{-1}(E)$ contains J_l for each $l \in \mathbb{N}$.

Claim 3. $J_0 \subseteq \varphi^{-1}(G)$.

Since $y_0 \in T$, if $t \in [\frac{1}{3}, 1]$, then $\varphi(t,0) \in (X \times T) - \{(p,q)\} \subseteq G$. This and Claim 1 show that J_0 is a subset of $\varphi^{-1}(G)$.

Claim 4. $J_l(\frac{1}{3}) = J_l \cap \varphi^{-1}(G)$ for each $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$ be arbitrary. From Claim 1, it follows that $J_l(\frac{1}{3}) \subseteq J_l \cap \varphi^{-1}(G)$. Now, let $(t,u) \in J_l$ be such that $\varphi(t,u) \in G$. The inclusion $y_l \in \Gamma_q^Y$ and (3.8.1) imply that $\varphi(t,u) \in \{z\} \times Y$. Hence, by (3.8.1), $t \leq \frac{1}{3}$ and so $J_l \cap \varphi^{-1}(G)$ is contained in $J_l(\frac{1}{3})$.

Finally, from claims 1 and 2, it follows that $J_0(\frac{1}{3}) \cup \bigcup_{l \in \mathbb{N}} J_l \subseteq \varphi^{-1}(E)$. So, since $J_0(\frac{1}{3}) \cup \bigcup_{l \in \mathbb{N}} J_l$ is a dense connected subset of F_H, we infer that $\varphi^{-1}(E)$ is connected. On the other hand, claims 1, 3 and 4 guarantees that $\varphi^{-1}(G) = J_0 \cup \bigcup_{l \in \mathbb{N}} J_l(\frac{1}{3})$ is connected. Then φ is monotone with respect to $\mathcal{U}$.

We have that $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$, $(0,0) \in \varphi^{-1}(\bigcap \mathcal{U})$ and $\mathcal{U}$, F_H, φ and $\langle (1, \frac{1}{k}), (1,0) \rangle \in \mathbb{S}(F_H)$ satisfy the required properties.

In conclusion, Z is a $\mathcal{U}-$Maya space.

4. Main Results

All results in this section together give the classification of points that make a hole in the product of two smooth dendroids.

Each corollary below can be proved using similar arguments of the proof of the previous theorem respectively.

Theorem 4.1. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and let $q \in Y$. If $v_X \in E(X)$, then (v_X, q) does not make a hole in $X \times Y$.*

Proof. Our assumption $v_X \in E(X)$ guarantees that $X \in \Delta_X(v_X)$. So, applying Lemma 3.10 we obtain that $X \times Y - \{(v_X, q)\}$ has property (b). Invoke Theorem 2.3 to prove that $X \times Y - \{(v_X, q)\}$ is unicoherent.

Corollary 4.2. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and let $p \in X$. If $v_Y \in E(Y)$, then (p, v_Y) does not make a hole in $X \times Y$.*

Theorem 4.3. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and let $(p, q) \in X \times Y$. If $p \in Ncut(X) - E(X)$ and $q \in Y - E(Y)$, then (p, q) does not make a hole in $X \times Y$.*

Proof. In light of Corollary 3.7, we need to prove that there exists a covering $\mathcal{U}$ of $(X \times Y) - \{(p, q)\}$ such that $(X \times Y) - \{(p, q)\}$ is $\mathcal{U}$-covered with respect property (b) and $(X \times Y) - \{(p, q)\}$ is a $\mathcal{U}$-Maya space.

Set $Z = X \times Y - \{(p, q)\}$ and fix $z \in v_X p - \{v_X, p\}$. For each $T \in \Delta_Y(q)$, let $U(T) = ((X \times (\Gamma_q^Y \cup T)) - \{(p, q)\}) \cup (\{z\} \times Y)$. Consider $\mathcal{U} = \{U(T) : T \in \Delta_Y(q)\}$. Notice that $\mathcal{U}$ is a covering of Z and, by Lemma 3.13, each element of $\mathcal{U}$ has property (b).

Let us argue that Z is $\mathcal{U}-$covered with respect to property (b). Consider $M = \{z\} \times Y$. Let $T \in \Delta_Y(q)$ be arbitrary. We have that $M \cap U(T) = M$ is connected. Now, notice that if $T_1, T_2 \in \Delta_Y(q)$, then $U(T_1) \cap U(T_2) \neq \emptyset$. Assume that $T_1 \neq T_2$. Define $L(U(T_1), U(T_2)) = ((X \times \Gamma_q^Y) - \{(p, q)\}) \cup (\{z\} \times Y)$. For sake of simplicity, L will represent to $L(U(T_1), U(T_2))$. Observe that L is a connected subset of Z and Lemma 3.12 ensures that L has property (b) . Moreover these sets satisfy: $U(T_1) \cap U(T_2) = L \cap U(T_1) = L \cap U(T_2) = L$ and $L \cap M = M$ are connected, and $(U(T_1) \cap M) \cup (U(T_2) \cap M) = M = L \cap M$. Thus, L fulfilling the conditions in the definition. Hence, Z is $\mathcal{U}-$covered with respect to property (b).

Now, in order to prove that Z is a $\mathcal{U}-$Maya space, let $(\langle (x_k, y_k) \rangle, (x_0, y_0)) \in \mathbb{S}(Z)$ be arbitrary. Taking subsequences, if it is necessary, by Lemma 3.4, we may assume that for each $k \in \mathbb{N}$ there exists $T_k \in \Delta_Y(q)$ satisfying that $y_k \in T_k$ and we only consider the following cases .

Case I. $y_0 \neq q$ and $z \in v_X x_k$ for every $k \in \mathbb{N}$.

Then we may assume that each $y_k \neq q$. Let $T_0 \in \Delta_Y(q)$ be such that $y_0 \in T_0$. Consider $\mathcal{V} = \{U(T_k) : k \in \mathbb{N} \cup \{0\}\}$. In light of (3.8.4), we may consider the mapping $g_{\Omega_z^X}$ and $g_{\Omega_q^Y}$. Define $\varphi : F_H \to Z$ by

$$\varphi(t,u) = \begin{cases} (z, g_{\Omega_q^Y}(y_l, 2t)) & \text{if } (t,u) \in J_l \text{ and } t \leq \frac{1}{2} \\ (g_{\Omega_z^X}(x_l, 2t-1), y_l) & \text{if } (t,u) \in J_l \text{ and } \frac{1}{2} \leq t \end{cases}$$

to get a map. Observe that $(0,0) \in \varphi^{-1}(\bigcap \mathcal{V})$. Now, we are going to prove that $\varphi^{-1}(U(T_k))$ is connected for each $k \in \mathbb{N} \cup \{0\}$. Let $k \in \mathbb{N} \cup \{0\}$ be arbitrary. Set $A = \{l \in \mathbb{N} \cup \{0\} :$ either $y_k \in T_l$ or $x_k = z\}$ and $B = \{l \in \mathbb{N} \cup \{0\} : y_k \notin T_l$ and $x_k \neq z\}$.

Claim 1. $\bigcup\limits_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{2}) \subseteq \varphi^{-1}(U(T_k))$.

Observe that if $(t,u) \in \bigcup\limits_{l \in \mathbb{N} \cup \{0\}} J_l$ is such that $t \leq \frac{1}{2}$, then $\varphi(t,u) \in \{z\} \times Y \subseteq U(T_k)$. Hence, we obtain that $J_l(\frac{1}{2}) \subseteq \varphi^{-1}(U(T_k))$ for each $l \in \mathbb{N} \cup \{0\}$.

Claim 2. $\bigcup\limits_{l \in A} J_l \subseteq \varphi^{-1}(U(T_k))$.

From Claim 1, it follows that $J_l(\frac{1}{2}) \subseteq \varphi^{-1}(U(T_k))$ for each $l \in A$. Now, let $l \in A$ be arbitrary and let $(t,u) \in J_l$ be such that $t \geq \frac{1}{2}$. If $y_l \in T_k$, by (3.8.5), we have that $\varphi(t,u) \in (X \times T_k) - \{(p,q)\} \subseteq U(T_k)$. Under the assumption $x_k = z$, by the definition of $g_{\Omega_z^X}$, we obtain that $\varphi(t,u) \in \{z\} \times Y \subseteq U(T_k)$. So, the inclusion $J_l \subseteq \varphi^{-1}(U(T_k))$ holds.

Claim 3. $J_l(\frac{1}{2}) = J_l \cap \varphi^{-1}(U(T_k))$ for each $l \in B$.

Let $l \in B$ be arbitrary. The inclusion $J_l(\frac{1}{2}) \subseteq J_l \cap \varphi^{-1}(U(T_k))$ is guaranteed by Claim 1. Next, let $(t,u) \in J_l$ be such that $\varphi(t,u) \in U(T_k)$. Since $y_k \notin \Gamma_q^Y \cup T_l$, we obtain that $\varphi(t,u) \in \{z\} \times Y$. From our assumption $x_k \neq z$ and (3.1.4), it follows that $t \not> \frac{1}{2}$. So, $(t,u) \in J_l(\frac{1}{2})$.

From claims 1, 2 and 3, we infer that $\varphi^{-1}(U(T_k)) = \left(\bigcup\limits_{l \in A} J_l\right) \cup \left(\bigcup\limits_{l \in B} J_l(\frac{1}{2})\right)$ is connected. Therefore, φ is monotone with respect to $\mathcal{V}$.

Finally, notice that $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$ and so, $\mathcal{V}$, F_H, φ and $\langle (1, \frac{1}{k}), (1,0) \rangle \in \mathbb{S}(F_H)$ satisfy the required properties.

Case II. $y_0 \neq q$ and $z \notin v_X x_k$ for each $k \in \mathbb{N}$.

From our assumption $y_0 \in Y - \{q\}$, we may assume that $\{y_k : k \in \mathbb{N}\} \subseteq Y - \{q\}$. Let $T_0 \in \Delta_Y(q)$ be such that $y_0 \in T_0$. Consider $\mathcal{V} = \{U(T_k) : k \in \mathbb{N} \cup \{0\}\}$. In light of (3.8.5), we can consider the mapping $g_{\Omega_q^Y}$. Define $\varphi : F_H \to Z$ by

$$\varphi(t,u) = \begin{cases} (z, g_{\Omega_q^Y}(y_l, 3t)), & \text{if } (t,u) \in J_l \text{ and } t \leq \frac{1}{3}, \\ (g_X(z, 2-3t), y_l), & \text{if } (t,u) \in J_l \text{ and } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ (g_X(x_l, 3t-2), y_l), & \text{if } (t,u) \in J_l \text{ and } \frac{2}{3} \leq t. \end{cases}$$

Then φ is a map. Let us show that φ is monotone with respect to $\mathcal{V}$. Let $k \in \mathbb{N} \cup \{0\}$ be arbitrary. Set $A = \{l \in \mathbb{N} \cup \{0\} : y_l \in T_k\}$ and $B = \{l \in \mathbb{N} \cup \{0\} : y_l \notin T_k\}$. We are going to prove the following claims.

Claim 1. $\bigcup_{l \in \mathbb{N} \cup \{0\}} J_l(\frac{1}{3}) \subseteq \varphi^{-1}(U(T_k))$.

If $(t,u) \in \bigcup_{l \in \mathbb{N} \cup \{0\}} J_l$ is such that $t \leq \frac{1}{3}$ and by (3.8.5), then $\varphi(t,u) \in \{z\} \times Y \subseteq U(T_k)$. Hence, $J_l(\frac{1}{3}) \subseteq \varphi^{-1}(U(T_k))$ for each $l \in \mathbb{N} \cup \{0\}$.

Claim 2. $\bigcup_{l \in A} J_l \subseteq \varphi^{-1}(U(T_k))$.

Let $(t,u) \in \bigcup_{l \in A} J_l$ be arbitrary. In light of Claim 1, we only need to assume that $t \geq \frac{1}{3}$. Then $\varphi(t,u) \in (X \times T_k) - \{(p,q)\} \subseteq U(T_k)$. So, $(t,u) \in \varphi^{-1}(U(T_k))$.

Claim 3. $J_l(\frac{1}{3}) = J_l \cap \varphi^{-1}(U(T_k))$ for each $l \in B$.

Let $l \in B$ be arbitrary. First, let $(t,u) \in J_l \cap \varphi^{-1}(U(T_k))$ be arbitrary. The condition $y_l \notin \Gamma_q^Y \cup T_k$ implies that $\varphi(t,u) \in \{z\} \times Y$. Now, since $z \notin v_X x_l$ and (3.1.5) holds, we have that $t \not> \frac{1}{3}$. Thus, $J_l \cap \varphi^{-1}(U(T_k)) \subseteq J_l(\frac{1}{3})$. The inclusion $J_l(\frac{1}{3}) \subseteq J_l \cap \varphi^{-1}(U(T_k))$ follows from Claim 1.

Thus, in light of claims 1, 2 and 3, we have that $\varphi^{-1}(U(T_k)) = \left(\bigcup_{l \in A} J_l\right) \cup \left(\bigcup_{l \in B} J_l(\frac{1}{3})\right)$ is connected. So, φ is monotone with respect to $\mathcal{V}$.

Observe that $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$ and $(0,0) \in \varphi^{-1}(\bigcap \mathcal{V})$. Then $\mathcal{V}$, F_H, φ and $\langle (1, \frac{1}{k}), (1,0) \rangle \in \mathbb{S}(F_H)$ fulfil all our requirements.

Case III. $y_0 = q$.

Then $x_0 \neq p$ and so we may assume that each $x_l \neq p$. Consider $\mathcal{V} = \{U(T_k) : k \in \mathbb{N}\}$. Observe that $\{(x_k, y_k) : k \in \mathbb{N} \cup \{0\}\} \subseteq \bigcup \mathcal{V}$. Define $\varphi : F_H \to Z$ by

$$\varphi(t,u) = \begin{cases} (g_X(x_l, 2t), v_Y), & \text{if } (t,u) \in J_l \text{ and } t \leq \frac{1}{2}, \\ (x_l, g_Y(y_l, 2t-1)), & \text{if } (t,u) \in J_l \text{ and } \frac{1}{2} \leq t, \end{cases}$$

to get a map.

Now, we shall prove that φ is monotone with respect to $\mathcal{V}$. Let $k \in \mathbb{N}$ be arbitrary. Set $A = \{l \in \mathbb{N} : \text{ either } y_l \in T_k \text{ or } x_l = z\}$ and $B = \{l \in \mathbb{N} : y_l \notin T_k \text{ and } x_l \neq z\}$. For each $l \in \mathbb{N}$, let $e_l \in [\frac{1}{2}, 1]$ be the unique point such that $g_Y(y_l, 2e_l - 1) = q$ (see (3.1.6)). Let us show the following claims.

Claim 1. $J_0 \subseteq \varphi^{-1}(U(T_k))$.

If $t \in I$, then $\varphi(t, 0) \in (X \times \Gamma_q^Y) - \{(p, q)\} \subseteq U(T_k)$. This proves that J_0 is a subset of $\varphi^{-1}(U(T_k))$.

Claim 2. $\bigcup\limits_{l \in A} J_l \subseteq \varphi^{-1}(U(T_k))$.

Let $(t,u) \in \bigcup\limits_{l \in A} J_l$ be arbitrary. The inclusion $(t,u) \in J_l(\frac{1}{2})$ and the definition of φ guarantee that $\varphi(t,u) \in (X \times \Gamma_q^Y) - \{(p,q)\} \subseteq U(T_k)$. Now, since either $y_l \in T_k$ or $x_l = z$, if $(t,u) \in J_l$ is such that $t \geq \frac{1}{2}$, then either $\varphi(t,u) \in (X \times (\Gamma_q^Y \cup T_k)) - \{(p,q)\}$ or $\varphi(t,u) \in \{z\} \times Y$. Thus, $J_l \subseteq \varphi^{-1}(U(T_k))$ for each $l \in A$.

Claim 3. $J_l(e_l) = J_l \cap \varphi^{-1}(U(T_k))$ for each $l \in B$.

Let $l \in B$ be arbitrary. First, if $(t,u) \in J_l(e_l)$, then $\varphi(t,u) \in (X \times \Gamma_q^Y) - \{(p,q)\} \subseteq U(T_k)$. This shows that $J_l(e_l)$ is contained in $J_l \cap \varphi^{-1}(U(T_k))$. Second, let $(t,u) \in J_l$ be such that $\varphi(t,u) \in U(T_k)$. Our assumptions $y_l \notin T_k$ and $x_l \neq z$ imply that $\varphi(t,u) \in (X \times \Gamma_q^Y) - \{(p,q)\}$. Assume that $t \geq \frac{1}{2}$. Then $\varphi(t,u) \in \{x_l\} \times g(\{y_l\} \times [0, 2e_l, 1])$. By (3.1.3), we infer that $t \leq e_l$. Thus, $(t,u) \in J_l(e_l)$.

So, from claims 1, 2 and 3, we deduce that $\varphi^{-1}(U(T_k)) = \left(\bigcup_{l\in A} J_l\right) \cup \left(\bigcup_{l\in B} J_l(e_l)\right)$ is connected. Thus, φ is monotone with respect to $\mathcal{V}$.

Observe that $\varphi(1, \frac{1}{k}) = (x_k, y_k)$ for all $k \in \mathbb{N}$. In conclusion, $\mathcal{V}$, F_H, φ and $\langle(1, \frac{1}{k}), (1, 0)\rangle \in \mathbb{S}(F_H)$ satisfy the required properties.

Therefore, Z is a $\mathcal{U}-$Maya space.

Corollary 4.4. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and let $(p, q) \in X \times Y$. If $p \in X - E(X)$ and $q \in Ncut(Y) - E(Y)$, then (p, q) does not make a hole in $X \times Y$.*

Theorem 4.5. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and let $(p, q) \in X \times Y$. If either $p \in E(X) - \{v_X\}$ or $q \in E(Y) - \{v_Y\}$, then (p, q) does not make a hole in $X \times Y$.*

Proof. Set $Z = (X \times Y) - \{(p, q)\}$. To show that Z is unicoherent, by Proposition 2.2 and Theorem 2.3, it suffices to verify that Z is contractible.

Define $\Psi : Z \times I \to Z$ by

$$\Psi((x, y), t) = (g_X(x, t), g_Y(y, t))$$

for each $((x, y), t) \in Z \times I$. To check that Ψ is well defined, let $((x, y), t) \in Z \times I$ be arbitrary. Suppose that $\Psi((x, y), t) = (p, q)$. Then $g_X(x, t) = p$ and $g_Y(y, t) = q$.

Suppose $p \in E(X) - \{v_X\}$. By the definition of g_X, we obtain that $p \in v_X x$ and, our assumption implies $p = x$. Hence, $g_X(p, t) = p$. Thus, by (3.1.5), the equalities $t = 1$ and $y = q$ hold. So $(x, y) = (p, q) \notin Z$, a contradiction. We conclude that $\Psi((x, y), t) \in Z$ and Ψ is well defined.

The continuity of Ψ follows from the fact that g_X and g_Y are continuous (see (3.1.2)). Finally, using (3.1.4) and (3.1.5), it can be proved that $\Psi((x, y), 1) = (x, y)$ and $\Psi((x, y), 0) = (v_X, v_Y)$ for each $(x, y) \in Z$. We conclude that Z is contractible.

Theorem 4.6. *Let X and Y be continua such that $X \times Y$ is unicoherent and let $(p,q) \in X \times Y$. If $(p,q) \in Cut(X) \times Cut(Y)$, then (p,q) makes a hole in $X \times Y$.*

PROOF. Since $X - \{p\}$ is not connected, there exist non-degenerate subcontinua H and G of X such that $X = H \cup G$ and $H \cap G = \{p\}$. Notice that $(H \times Y) - \{(p,q)\}$ and $(G \times Y) - \{(p,q)\}$ are connected closed subsets of $(X \times Y) - \{(p,q)\}$ whose union is $(X \times Y) - \{(p,q)\}$ and their intersection is homeomorphic to $Y - \{q\}$ which is not connected. This proves that $(X \times Y) - \{(p,q)\}$ is not unicoherent.

Classification

Theorem 4.7. *Let (X, v_X) and (Y, v_Y) be smooth dendroids and let $(p,q) \in X \times Y$. Then (p,q) makes a hole in $X \times Y$ if only if $(p,q) \in Cut(X) \times Cut(Y)$.*

PROOF. Let $(p,q) \in X \times Y$ be such that (p,q) makes a hole in $X \times Y$. First, notice that $X = E(X) \cup Cut(X) \cup Ncut(X)$, $Y = E(Y) \cup Cut(Y) \cup Ncut(Y)$, $E(X) \subseteq Ncut(X)$ and $E(Y) \subseteq Ncut(Y)$. Second, since (p,q) makes a hole in $X \times Y$, by Theorem 4.1, Corollary 4.2 and Theorem 4.5, we infer that $p \notin E(X)$ and $q \notin E(Y)$. So, we deduce that $p \in Cut(X) \cup (Ncut(X) - E(X))$ and $q \in Cut(Y) \cup (Ncut(Y) - E(Y))$. From Theorem 4.3 and Corollary 4.4, it follows that $p \in Cut(X)$ and $q \in Cut(Y)$.

The converse follows from Theorem 4.6.

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